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Algebraic Structures in Mathematical Music Theory.

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Algebraic Structures in Mathematical Music Theory[1]

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In this brief participation I will talk about the algebraic structures that appear in Mathematical Music Theory. This expository dialog is for a general audience, and for mathematicians or musicians as well. Concrete examples will be given. It will be said that, to a certain extent, these structures classify mathematics. Also, I will mention general concept architectures such as denotators and forms. Finally, I will expose the mathematical view of objects, structures and concepts discussed with Guerino Mazzola some years ago. I will not use any written mathematical notation and spell it with only words. A very difficult task but I will try. I will assume some basic knowledge in mathematics and music but if you do not know it, please do not discourage, and keep going.

I am going to refer to the algebraic structures part, in Mathematical Musicology or Mathematical Music Theory. I will mention for a general audience and for mathematicians or musicians in particular, examples where algebraic structures appear.

You can consult my Modern Algebra book for a formal and precise definitions of the algebraic structures I will mention. The study of algebraic structures or algebraic systems are the objects of study of what is called Abstract Algebra or Modern Algebra.

Remember that a binary operation on a set is a function whose domain is the binary cartesian product of it with itself, with codomain the same set. A ternary operation would be a function whose domain is a ternary product of the set with the set as codomain.

An algebraic structure or algebraic system is a set together with one or more n-ary operations defined on it which could satisfy certain axioms or properties.

We consider several algebraic structures. For example:

A magma or groupoid is just a nonempty set with a binary operation.

If a magma or groupoid has the associative property it is called a ***semigroup***.

If a semigroup satisfies the identity element property it is called a ***monoid***.

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[1]Text corresponding to the author´s participation in the ROUND-TABLE 3 of the 7th MusMat International Conference. Rio de Janeiro University. (20/10/2022).

A ***group*** is a nonempty set together with a binary operation that is associative, has an identity element and has an inverse for every element in the set. These means that a group is a monoid that satisfies the inverse property.

A group is called ***abelian*** if every element commutes with each other.

A *ring* **is a nonempty set together with two binary operations such that it is an abelian group under one such binary operation, a semigroup under the other binary operation and satisfies an adequate distribution law.**

A ***field*** is a commutative ring with division. Please see my book on Modern Algebra for details.

Remember from a Linear Algebra course that a ***vector space*** is a nonempty set with one binary operation which is an abelian group under it, together with a scalar multiplication, that is an action of a field on the set that satisfies adequate distributive actions and the action of the identity element of the field leaves invariant the element on which it acts.

If instead of a field, you generalize the previous definition to the action of a ring on the set, you get the important concept of ***module over a ring***.

The algebraic structure called ***algebra over a ring*** is a nonempty set that simultaneously is a ring and a module over a ring.

As a matter of fact, the Classification of Mathematics partly follows the structure associated with a set. The Mathematics Subject Classification System, MSC 2020, fulfills this.

For example, we have the following areas of Mathematics:

08 designates General Algebraic Systems

13 is Commutative Algebra. That is the study of commutative rings.

14 is Algebraic Geometry. That is, the study of curves, surfaces, and algebraic varieties as solutions of polynomial equations, vaguely speaking.

15 is Linear Algebra, Multilinear Algebra and Matrix Theory. This area is the study of the specific properties of linear equations, vector spaces and matrices, generally speaking.

16 is Associative Rings and Algebras

17 is Nonassociative Rings and Algebras

18 is Category Theory and Homological Algebra. That is, the study of categories and functors, and the fundamental concept of all mathematics in the twentieth century, “homology”.

19 is K-theory. That is the study of the K-groups associated to a ring or category.

20 is Group Theory and generalizations.

22 is Topological groups, Lie groups.

55 is Algebraic Topology. This is the study of topological spaces using Abstract Algebra vaguely speaking.

A few examples of where algebraic structures or algebraic systems appear or are used in Mathematical Music Theory or Mathematical Musicology are:

Groups are used in composition and analysis of dodecaphonic series.

The set $\{1_{OP}, R, I, RI\}$ where R is the retrograde, I is the inversion and RI is the retrograde inversion IR, O denotes the onsets and P the pitches define a group of four elements called the Klein 4-group. It is generated by R and I and has two nontrivial subgroups.

This fact can be generalized and gives rise to a general concatenation principle in music theory, namely that all groups that are important to music are in fact so because they admit sets of generators that are musically understandable.

Actions of a group in a set are very important in Mathematical Music Theory. They could be free, transitive, simply transitive, regular, etc. Simply transitive actions in groups are in correspondence with Lewin´s Generalized Interval System.

Group actions are very frequent in music theory. They allow us to define in a precise way chords, transposition classes of chords, diminished triads, mayor triads, etc.

Also, cyclic groups of higher order are used for microtonal compositions. Sometimes the diatonic scale 0, 2, 4, 5, 7, 9, 11 is modeled in Z_7 as if the tonal distances were equal. Z_{12} is a very important cyclic group in Music Theory.

Another example of an important group in Mathematical Music Theory is the third torus which is the finite abelian group $Z_3 \times Z_4$.

The ring algebraic structures that appear in Music Theory are Z_3, Z_4, Z_{12}, and the product ring $Z_3 \times Z_4$.

Modules over the rational or real numbers, that is, vector spaces over those fields are of very important use. They give us geometric representation of musical objects and are the general structures that realize what scores do. The geometric representation of musical objects as subsets of modules is not only precise and complete, but it also opens the action of groups of transformations to simplify creativity. Not only is a theoretical advantage, but it is also used in several music software. For example:

The UPIC tool of Iannis Xenakis automatically transformed two-dimensional drawings into sounds. Cool235. It was a composition machine that allows the composer to draw musical objects on an interactive graphical interface.

Guerino Mazzola´s PRESTO, was a geometric interface where any affine transformation can be performed in addition to direct drawing.

Also, Mazzola´s software RUBATO was developed as a place where Mathematical Music Theory has been implemented in a quite general mathematical framework of general concept architectures such as Denotators and Forms. Here, visualization and sonification can be performed in very general spaces. My former student, Mariana Montiel worked developing these concepts back in 1999 and 2000. She presented them in her Master´s thesis, me as her advisor at UNAM. I imagine she will talk about these in her participation in this round table.

All the algebraic structures mentioned till now share the fact that they are sets with certain structure and a way to relate them through what we call homomorfisms and a decent way to compose them. The bottom line of all these algebraic structures is what we call a category.

An example of one of the most important categories in Mathematical Music Theory is the category of local compositions in modules over a ring.

It happens in all science and art, that if you have powerful and sophisticated mathematical tools there are better chances of analysing, creating, modeling and performing musical phenomena.

I invite you to see the works of my former doctoral student, Octavio Alberto Agustín-Aquino where he used the concepts previously mentioned in Counterpoint. Also see the works by

Juan Sebastián Arias-Valero published in MusMat recently, where Algebraic Topology is present besides the concepts mentioned above.

To end up my intervention I want to share with you an extract of my email conversation with Guerino Mazzola which are worth mentioning. He agreed I mentioned them in public. I deleted the nonmathematical parts. So, here they are:

On Saturday, December 7, 2019, he wrote:

Dear Emilio,

I have a theory about the mathematics' evolution in the large shape:
There have been, to my understanding, three phases:

1. Mathematics of objects.
Here special objects, such as distinguished function spaces, are at the center of interest. To my mind the completion of this phase was the Cantor set theory, where any of the classical objects could be constructed as special sets.

2. Mathematics of structures.
Here the concrete objects are no longer of central interest, it is more the structures which matter, groups, rings, modules, etc. To my mind, the completion of this phase was the Eilenberg-Mac Lane theory of categories. Categories are the focus on structure types, no longer on concrete objects.

3. Mathematics of concepts.
Here, the focus is no longer on particular structure types, i.e., categories, but on the construction of mathematically powerful concepts to solve major problems. To my mind, it was above all Grothendieck and his followers, such as prominently Lawvere, who focused on conceptual invention and creativity. The solution of Weil conjectures by Deligne was precisely this type of finding the right concept of cohomology to tackle Weil's conjectures. The concept of a topos is precisely the conceptual solution of what set theory would look like without concrete sets, it is the dialectic resolution of embedding set theory in structural mathematics.

To my mind, we are now in this third phase, but we still don't have a general methodology, or even the consciousness of the urgency of understanding the shift from category theory to concept mathematics.

I would be honored if you could give me some thoughts about these ideas.

All the best, Guerino.

Same Saturday, December 7, 2019.

Dear Guerino:

I agree with your vision on mathematics evolution. Let me comment that since last century the Classification of Mathematics was and still is mainly about structures defined on sets.

I remember talking with Christopher Soule (last century) that Algebraic K-Theory was like a Mexican pyramid. A big building without a top! This is because it was created to solve Serre's Conjecture and that its solution was obtained by Quillen and Suslin independently and without using K-Theory. But this fact does not reduce the power of the field.

I read long ago all Atiyah papers where he expresses his panoramic mathematical viewpoint. He expresses that the future mathematics would be the one that explains the human brain! At some point he said that innovation in mathematics takes place when the concepts pass through many mathematicians' brains and get compacted and explained in a clearer way. He asks if the big mathematical building will drop and squash us. The answer is no. Big pieces of mathematics will be subsumed or incorporated as special cases of new concepts.

Also, I remember Jacob Palis words when I told him back in 2001 about Mathematical Music Theory. He was the president of the International Mathematical Union (IMU) at that time, and I was the president of the Sociedad Matemática Mexicana. I invited him to the Copacabana Palace to hear a concert. He did not know I was a concert pianist and to his surprise, instead of sitting with him in the audience, he saw me coming out from backstage and sat at the piano. I played Rachmaninoff's Second piano concerto as soloist of the Rio de Janeiro Philharmonic Orchestra. He was very enthusiastic and happy. We went for dinner, and he told me the same idea as the one Grothendieck told you: "That is the mathematics of the future!". So, we can see here that new mathematics are created by means of the music. The same way as it happened with Physics that motivated new mathematics long time ago and continues to do so now.

I think that our very good (admired by us) friend Fernando Zalamea would have a very interesting point of view about mathematics evolution since he has thought and studied profoundly so much mathematics.

Please receive my best wishes!!! Emilio.

On Sunday December 8, 2019.

Dear Emilio, thank you so much for your very inspiring answer.
As to Atiyah's claim: "he expresses that the future mathematics would be the one that explain the human brain!" I somehow agree. That's what I would call the conceptual mathematics because the brain is working with concepts, more and more in mathematics, too.

In a recent conference about creativity and computers, I proposed that creativity is always an extension of a system of signs, a semiotic. Therefore, if computers should become creative, we should have a computerized/software representation of semiotic systems.

For this reason, I have now developed a mathematical theory of semiotics, that would be the first step to a software for creativity.

I have now published this theory in the Journal of Mathematics and Music under the title "Functorial Semiotics for Creativity". I add this paper for you here.

I should also include Fernando in this email, so I include my previous text for him, ok?

All the best, Guerino.

What we can get from these ideas is that there is a revolution going on in the mathematics we are using or working with. New mathematics are needed for specific unsolved problems. A few days ago, Guerino told me that he is working now on a General Theory of Concepts, including sets, categories, etc. I can consider this as one of the states of our art. Quite a challenging project.

To finish this intervention, let me express once more that for me, Mathematics is a Fine Art, the purest of them, who has the gift of being the most precise and the precision of Science.

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